A Note on Cyclic Gradients

Dan Voiculescu

To the memory of Gian-Carlo Rota

The cyclic derivative was introduced by G.-C. Rota, B. Sagan and P. R. Stein in [3] as an extension of the derivative to noncommutative polynomials. Here we show that there are simple necessary and sufficient conditions for an n-tuple of polynomials in n noncommuting indeterminates to be a cyclic gradient (see Theorem 1) and similarly for a polynomial to have vanishing cyclic gradient (see Theorem 2). Our interest in cyclic gradients stems from free probability theory and random matrices (see the Remark at the end) [1],[2],[4],[5],[6]. This note should also reduce the paucity of results on cyclic derivatives in several variables pointed out in [3, page 73].

Let $K_{\langle n\rangle} = K\langle X_1,\dots,X_n\rangle$ be the ring of polynomials in noncommuting indeterminates $X_1,\dots,X_n$ with coefficients in the field K of characteristic zero. The partial generalized difference quotients are the derivations

$$\partial_j : K_{\langle n\rangle} \to K_{\langle n\rangle} \otimes K_{\langle n\rangle}$$

such that $\partial_j X_k = 0$ if $j \neq k$ and $\partial_j X_j = 1 \otimes 1$. The $\otimes$ here is over K and $K_{\langle n\rangle} \otimes K_{\langle n\rangle}$ is given the bimodule structure such that $a(b\otimes c) = ab\otimes c$, $(b\otimes c)d = b\otimes cd$.

The partial cyclic derivatives are then

$$\delta_j = \tilde{\mu}\circ\partial_j : K_{\langle n\rangle} \to K_{\langle n\rangle}$$

where $\tilde{\mu}(a\otimes b) = ba$.

We shall denote by $N : K_{\langle n\rangle} \to K_{\langle n\rangle}$ the "number operator", i.e. the linear map so that $N1 = 0$, $NX_{i_1}\dots X_{i_k} = kX_{i_1}\dots X_{i_k}$. Also, $CK_{\langle n\rangle}$ will denote the cyclic subspace, i.e. the vector subspace spanned by all cyclic symmetrizations of monomials

$$CX_{i_1}\dots X_{i_p} = \sum_{1\le j\le p} X_{i_{j+1}}\dots X_{i_p}X_{i_1}\dots X_{i_j}\ , \qquad p\geq 1 \ \text{ and } \ C1=0$$

(the constants are not in the cyclic subspace).

Theorem 1 *Let $P_1, \dots, P_n \in K_{\langle n \rangle}$. The following conditions are equivalent:*

(i) *there is $P \in K_{\langle n \rangle}$ such that $\delta_j P = P_j$ $(1 \le j \le n)$.*

(ii) $\sum\limits_{1 \le j \le n} [X_j, P_j] = 0$.

(iii) $\sum\limits_{1 \le j \le n} X_j P_j \in CK_{\langle n \rangle}$.

(iv) $\delta_k \left(\sum\limits_{1 \le j \le n} X_j P_j \right) = (N + I) P_k$. *(Here I denotes the identity map of $K_{\langle n \rangle}$ to itself.)*

Proof. It is easily seen that it suffices to prove the theorem for homogeneous $P_1, \dots, P_n$ of the same degree, i.e. we may assume $NP_j = sP_j$ $(1 \le j \le n)$ for some $s \ge 0$. Also the case of constants being obvious we will concentrate on $s \ge 1$.

(i) $\Rightarrow$ (ii) To check that $\sum\limits_{1 \le j \le n} [X_j, \delta_j P] = 0$ it suffices to do so when P is a monomial $X_{i_0} \dots X_{i_s}$. Then

$$\delta_j P = \sum_{i_p = j} X_{i_{p+1}} \dots X_{i_s} X_{i_0} \dots X_{i_{p-1}}$$

so that

$$[X_j, \delta_j P] = \sum_{i_p = j} \left(X_{i_p} \dots X_{i_s} X_{i_0} \dots X_{i_{p-1}} - X_{i_{p+1}} \dots X_{i_s} X_{i_0} \dots X_{i_p} \right)$$

and hence

$$\sum_{1 \le j \le n} [X_j, \delta_j P] = \sum_{1 \le p \le s} \left(X_{i_p} \dots X_{i_s} X_{i_0} \dots X_{i_{p-1}} - X_{i_{p+1}} \dots X_{i_s} X_{i_0} \dots X_{i_p} \right) = 0 \ .$$

(ii) $\Rightarrow$ (iii) Let

$$P_j = \sum_{i_1, \dots, i_s} c^j_{i_1 \dots i_s} X_{i_1} \dots X_{i_s}$$

The coefficient of $X_{i_0} \dots X_{i_s}$ in $\sum\limits_{1 \le j \le n} [X_j, P_j]$ is

$$c^{i_0}_{i_1 \dots i_s} - c^{i_s}_{i_0 \dots i_{s-1}} \ .$$

Hence (ii) gives $c^{i_0}_{i_1 \dots i_s} = c^{i_s}_{i_0 \dots i_{s-1}}$. On the other hand, if $c_{i_0 \dots i_s}$ denotes the coefficient of $X_{i_0} \dots X_{i_s}$ in $\sum\limits_{i \le j \le n} X_j P_j$, clearly $c_{i_0 \dots i_s} = c^{i_0}_{i_1 \dots i_s}$, so that (ii) implies $c_{i_0 \dots i_s} = c_{i_s i_0 \dots i_{s-1}}$, i.e. cyclicity.

(iii) $\Rightarrow$ (iv) As before, let $c^j_{i_1\ldots i_s}$ and $c_{i_0\ldots i_s}$ denote the coefficients of P_j and $\sum_j X_j P_j$ respectively. Then $c^{i_0}_{i_1\ldots i_s} = c_{i_0\ldots i_s}$ and the cyclicity condition gives $c_{i_0\ldots i_s} = c_{i_s i_0\ldots i_{s-1}}$. We have

$$\begin{aligned}
&\delta_k\left(\sum_k X_j P_j\right)\\
&= \sum_{i_0\ldots i_s}\sum_{\{r:i_r=k\}} c_{i_0\ldots i_s} X_{i_{r+1}}\ldots X_{i_s}X_{i_0}\ldots X_{i_{r-1}}\\
&= \sum_{i_0\ldots i_s}\sum_{\{r:i_r=k\}} c^k_{i_{r+1}\ldots i_s i_0\ldots i_{r-1}} X_{i_{r+1}}\ldots X_{i_s}X_{i_0}\ldots X_{i_{r-1}}\\
&= \sum_{0\le r\le s}\ \sum_{i_0\ldots i_{r-1}i_{r+1}\ldots i_s} c^k_{i_{r+1}\ldots i_s i_0\ldots i_{r-1}} X_{i_{r+1}}\ldots X_{i_s}X_{i_0}\ldots X_{i_{r-1}} = (s+1)P_k\ .
\end{aligned}$$

(iv) $\Rightarrow$ (i) Since the P_j are homogeneous of the same degree and the field characteristic is zero, this is obvious. $\square$

There is also a simple description of the noncommutative polynomials with vanishing cyclic gradient.

Theorem 2 *We have*

$$\mathrm{Ker}\ \delta \ = \sum_{1\le k\le n}[X_k, K_{\langle n\rangle}] + \mathbb{C}1 \ = \ \mathbb{C}1 + [K_{\langle n\rangle}, K_{\langle n\rangle}] \ = \ \mathrm{Ker}\ C$$

Proof. (i) $\mathrm{Ker}\ \delta \subset \mathrm{Ker}\ C$. We have

$$Cp \ = \sum_{1\le j\le m} X_j \delta_j p \ = \ 0$$

(ii) Clearly,

$$\sum_{1\le k\le n}[X_k, K_{\langle n\rangle}] + \mathbb{C}1 \subset [K_{\langle n\rangle}, K_{\langle n\rangle}] + \mathbb{C}1$$

Also, since $1 \in \mathrm{Ker}\ C$ and $[X_{i_1}\ldots X_{i_r}, X_{i_{r+1}}\ldots X_{i_{r+s}}]$ is the difference of two cyclic permutations of $X_{i_1}\ldots X_{i_{r+s}}$, we have $[K_{\langle n\rangle}, K_{\langle n\rangle}] + \mathbb{C}1 \subset \mathrm{Ker}\ C$.

To see that $\mathrm{Ker}\ C \subset \sum_{1\le k\le n}[X_k, K_{\langle n\rangle}] + \mathbb{C}1$, remark that $\mathrm{Ker}\ C$ is spanned by homogeneous elements and that $Cp = 0$, where p is homogeneous of degree m iff p is a linear combination of differences $X_{i_1}\ldots X_{i_m} - X_{i_2}\ldots X_{i_m}X_{i_0}$.

(iii) To see that $\sum_{1 \le k \le n} [X_k, K_{\langle n \rangle}] + \mathbb{C}1 \subset \text{Ker } \delta$, it suffices to show that $[X_k, X_{i_1} \dots X_{i_s}] \in \text{Ker } \delta$. This is clearly so, since

$$[X_k, X_{i_1} \dots X_{i_s}] = X_k X_{i_s} \dots X_{i_s} - X_{i_1} \dots X_{i_s} X_k$$

and the cyclically equivalent elements $X_k X_{i_1} \dots X_{i_s}$, $X_{i_1} \dots X_{i_s} X_{i_k}$ have the same cyclic gradient. □

Putting together the two theorems, we have an exact sequence

$$0 \to [K_{\langle n \rangle}, K_{\langle n \rangle}] \to K_{\langle n \rangle} \xrightarrow{\delta} (K_{\langle n \rangle})^n \xrightarrow{\theta} K_{\langle n \rangle}$$

where $\theta((P_j)_{1 \le j \le n}) = \sum_j [X_j, P_j]$.

Remark. The motivation for this note is from free entropy and large deviations for random matrices. Let (M, τ) be a von Neumann algebra with normal faithful trace-state τ and $X_k = X_k^* \in M \quad (1 \le k \le n)$ which are algebraically free.

Let $\mathcal{J}_k = \mathcal{J}(X_k : \mathbb{C}\langle X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n \rangle)$ be the noncommutative Hilbert transforms defined in [4], in connection with free entropy. On the other hand, the upper bound for large deviations for n-tuples of random matrices found in [2] fits well with free entropy except for a term involving cyclic gradients and about which it is not known whether it is not actually zero. The precise question is, whether the n-tuple $(\mathcal{J}_k)_{1 \le k \le n}$ (when it exists) is a limit in 2-norm of cyclic gradients of polynomials in the noncommuting variables $X_1, \dots, X_n$? The theorem we proved here provides a partial affirmative answer:

> If $(\mathcal{J}_k)_{1 \le k \le n}$ are noncommutative polynomials in $X_1, \dots, X_n$, then there is a noncommutative polynomial P in $X_1, \dots, X_n$ such that $\mathcal{J}_k = \delta_k P \quad (1 \le k \le n)$.

Indeed, by Corollary 5.12 in [5] we have $\sum_k [\mathcal{J}_k, X_k] = 0$. Hence the commutator condition (ii) in the Theorem is satisfied.

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Department of Mathematics
University of California
Berkeley, California 94720–3840
dvv@math.berkeley.edu